Majority Voting and Single Peakedness

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Majority Rule is a method of decision making which is widely used and almost universally accepted as being "good", useful or fair. It is important, therefore, that we examine the underlying rationale for the acceptance of Majority Voting as a legitimate method of social choice and that we investigate the circumstances in which its use can be justified. In this latter context we shall largely be concerned with the issue of Single Peakedness. My approach will be to discuss, firstly, voting systems and the desirable properties they should have as collective choice mechanisms, and, secondly, the issue of Single Peakedness as it pertains to the successful formulation of social preferences.

A Social Welfare Function is a rule or process which, for each set of individual orderings of a group of alternative social states, selects a corresponding social ordering of the same social states. The first issue raised by this definition is whether or not it is rational to speak of society having preferences.

In the event that every person in a society prefers social state x to social state y it is generally accepted that society, as a unit, prefers x to y also. However in a case where a majority (i.e. over half) prefer x to y it is not at all so clear that society should prefer x to y. In particular, if the choice of x results in disutility to those who prefer y (the minority) greater than the majority's utility (in x) then the counter-argument that society prefers y to x because aggregate utility is higher might be valid.(*1) Another problem with accepting that the preferences of a majority should determine those of society, is that paradoxical cycles in society preference structures are obtainable. Clearly these two issues must be considered further in the development of a satisfactory Social Welfare Rule.

A number of different voting systems have been proposed which might be useful as Decision Rules. I intend to briefly outline some of these and then discuss their desirability as <u>Social Orders</u>.

A. Borda Count System:
This method involves the individual assigning values from 0 to n-1 to the n social states under review. For 3 alternative social states, an individual would assign the integers 0, 1 and 2 to the states. It therefore implicitly assumes that utility is cardinal because a score of 2, with twice the weighting of a score of 1, implies the individual prefers the former twice as much as the latter. However, despite this limitation, this system, and others with different weightings, do take account of intensity of preferences to a certain extent. It is not, in any case, a method of majority voting.(*2)

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*1 I do not wish to suggest that utility may be aggregated or analysed in this manner. My purpose is to show that a majority outcome is not unambiguously 'best' for society.

*2 The following preference structure will illustrate this:

Individual 1	x,	y,	z,	w.	x gets 6 points
Individual 2	x,	y,	w,	z.	y gets 7 points
Individual 3	y,	z,	w,	x.	z gets 3 points
	(3)	(2)	(1)	(0)	w gets 2 points

y wins despite the fact that a majority favours x. y might be thought of as the least-worst alternative.

B. First-Past-The-Post

This system is not strictly a Social Welfare Ordering because it does not give an ordering of preferences for Society. It does however select that state with the highest number of votes and is in common use (eg the UK elections to Parliament). The system is very susceptible to strategic voting.(*1)

C. Knock-Out System

This involves the elimination of the least preferred alternative and then proceeding to a new round of voting until there is one alternative left. Under certain circumstances this system is identical to the Proportional Representation method (used, for example, in the Republic of Ireland).(*2)

D. The Committee System

The method here is to compare pairs of alternatives. In particular, the alternative preferred by the majority from the first pairing is then run against another alternative and so on until a winner emerges. At small group level (eg up to 20 people) this method is often used. It is important because most political and economic decisions taken on a day-by-day basis are the result of this method.

We should note at this stage that B, C and D are all majority voting systems in the sense that if a majority of people prefer x to y then each of these systems will select x as preferable to y. A is not, however, a majority voting system in this sense (as mentioned in footnotes above).

The following properties are often considered to be desirable in a voting system or method of decision making.

1. If state x is pareto superior to state y then x should be chosen over y. Clearly A, B, C and D all satisfy this condition because there are no losers to vote against x and at least one gainer who votes for it.

2. Everybody's preferences are accorded equal weight or importance. This precludes dictatorship which many consider undesirable anyway. It is controversial because intensity of preference is ignored.(*3) For

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*1 To see how strategic voting is favoured take the following example:

i = 1 ... 5:	x,	y,	z.
i = 6 ... 9:	z,	x,	y.
i = 10, 11:	y,	z,	x.

The incentive to the last two individuals, if they know y is not to be chosen anyway, is to vote for z so that, a) their preferences will count in the selection, and, b) z will be chosen instead of x.

*2 The systems are the same except for the fact that individuals using the knock-out system may behave differently than their preferences suggest they should, because all of their preference structure is not revealed simultaneously as it is in the PR system.

*3 The issue of intensity of preferences is important because if we could incorporate them we would have "natural" winners which are so by virtue of being best for society in the pareto sense. It should also be noted that two types of intensity exist, one within individual preference structures between the alternatives and the other across individuals for a particular alternative. The former might be taken account of in the Borda System where it may be seen that despite a majority acceptance an alternative may be rejected. Condition 2 is a trade-off with the latter.

example, at the extreme it denies the right to a veto of a small (minority) group or even an individual. It is, nonetheless, generally considered acceptable by default, i.e. simply because usually no better suggestion as to weightings of votes can be made. (*1)

3. All logically possible preference structures of individuals should be admissable by the procedure i.e. the domain of the decision rule should be unrestricted. Thus any preferences, however unlikely and/or inconsistent they may appear, must be respected as legitimate.

   All of our voting systems agree with conditions 2 and 3.

4. Irrelevant alternatives should not influence the outcome between any two states. For example, a system choosing between states x and y should rank them similarly regardless of the inclusion or otherwise of any third alternative z which is irrelevant (in the sense of not being a substitute or a complement for either x or y). Each of the systems B, C and D fail to satisfy this condition if the domain is unrestricted. (*2)

5. A complete ordering of all alternatives for society should be the outcome and this ordering should be transitive so as to be meaningful and useful.

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*1 For example, intensity is taken account of by the EEC Council of Ministers in allowing a veto on affairs of vital national interest. This is to change.

*2 An example is as follows:

i = 1, 2	:	x,	y,	z.	x P y
i = 3	:	z,	x,	y.	x P y
i = 4, 5	:	y,	z,	x.	y P x

In a contest between x and y (z excluded) x wins. Including z yields the result in the committee system that z beats x and y beats z so y is chosen. Solutions to this problem find support in strong chairmen or strict agenda rules to prevent this type of voting cycle.

Under the first-past-the-post system with the following preferences,

i = 1, 2	:	z,	x,	y.	x P y
i = 3, 4	:	x,	y,	z.	x P y
i = 5, 6	:	y,	x,	z.	y P x
i = 7	:	y,	z,	x.	y P x

In a simple contest between x and y, x wins 4 to 3.

Including z we get

x	2
y	3
z	2

so y wins.

This might explain why there is discontent among the SDP/Liberal Alliance in the UK with the first-past-the-post system.

Of the four systems mentioned above, no two will always produce the same result for all possible domains so at best only one of them can be a useful decision rule. This is clear because the acceptance of two decision rules which might produce different outcomes would defeat the purpose of social choice.

In fact, it is not just the four systems which I mention here which fail to satisfy the conditions but, as Arrow's Theorem proves, no voting system exists which satisfies them. Thus one of the conditions must be foregone in order for society to arrive at a decision between alternatives.

If the emergence of a complete, transitive ordering of social states is the object of this exercise then condition 5 must remain. Various relaxations of the other conditions have been suggested. For example, a dictator might be allowed, or an end-state assigned by computer, or various other means would produce a social ordering. However the loss in consumer sovereignty would be great.

A different approach would be to see which, if any, of the conditions is a lesser constraint on the process than any of the others. The result of this line of investigation reveals that condition 3 of unrestricted domain only matters some of the time so we proceed further.

We can illustrate Arrow's theorem using the committee system of voting and an example of a preference ordering as follows:

$i = 1$ $x > y > z$
$i = 2$ $y > z > x$
$i = 3$ $z > x > y$

(">" = "is preferred to")
i refers to an individual.

In a contest

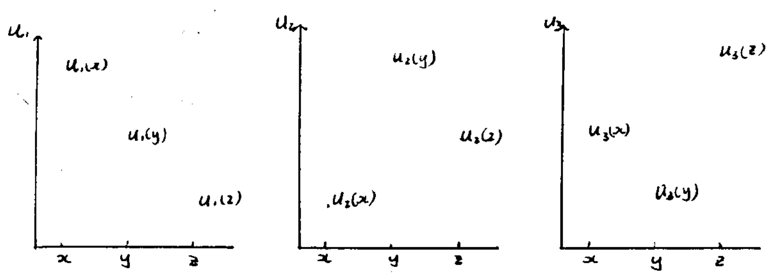
x vs. y => x wins
x vs. z => z wins
z vs. y => y wins

So x "Psoc" y "Psoc" z "Psoc" x etc.

("Psoc" = "is preferred by society to")

This particular preference configuration produces a meaningless result for society. I intend to show that if we prohibit inclusion of such a structure of preferences (known as Latin Square Design) we get a transitive social ordering.

If we show this preference structure in utility space it looks like this:

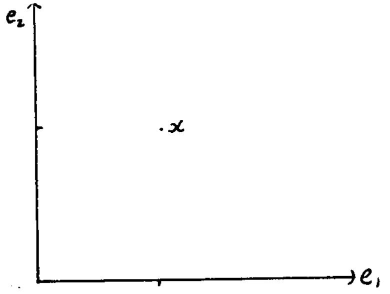


(No figures or values may be placed on the utilities above and the vertical axes in each diagram are not necessarily connected).

If the alternative social states under discussion are all located along a single dimension in space then there is a property of the orderings, called single peakedness, which, if it holds, makes majority voting (a la committee system) transitive at best and acyclic at worst. In terms of the utility diagrams above we may define single peakedness as requiring the existence of an ordering of end-states such that (for that ordering) each individual's utility function for the alternative end-states is unimodal (i.e. single peaked). It is important to emphasise that it is a property of the orderings not of the utility functions. Single peakedness precludes the Latin Square Design so some alternative is never worst, and, by ordering the alternatives such that this alternative is in the middle, we will get unimodal utility functions.

In terms of actual alternatives this is tantamount to saying "If i prefers x to y, then i prefers any point between x and y to y itself". For example, if x is £10,000 expenditure on roads and y is £20,000, then £15,000 or £18,000 are preferable to £20,000. Single peakedness of preferences would, therefore, seem to be a reasonable assumption to make in economics. The assumption simply states that those preference structures where the committee system is indecisive are so rare and unlikely that they may be ignored. Under Single Peakedness, therefore, majority voting constitutes a Social Welfare Ordering.

Often, however, a group must decide on two types of issue simultaneously and the alternatives may be located in two-dimension space.



Point x in this diagram represents levels of e1 and e2. If x is individual i's most preferred alternative then we may discuss the two-dimension analogue of single peakedness with respect to point x here. Point x, if preferences are single peaked should be enclosed by iso-utility lines which should be convex, continuous, and thin (i.e. each point on the line touches two and only two other points). Even with this assumption in the two-dimensional case, transitivity of preferences for society can only be guaranteed by other highly limiting assumptions. As an example of this consider a three-dimensional case of three individuals allocating society's wealth between them.

Distributions of wealth may be represented by vectors and imagine that the following alternatives are proposed:

A: (0, 1, 2)
B: (1, 2, 0)
C: (2, 0, 1)

A vs. B => B wins because two people better off
B vs. C => C wins because two people better off
C vs. A => A wins because two people better off

Thus two people can form a blocking coalition for any distribution. This scenario is quite realistic and illustrates that the multidimensional analogue of Single Peakedness is an untenable assumption.

In conclusion, therefore, it is clear that there is a serious problem in Social Choice Theory, namely that no satisfactory decision rule exists which possesses a given set of basic desirable properties. We have mentioned one example of a possible solution to this problem (i.e. single peakedness), whereby the committee system of majority voting may constitute a Social Welfare Ordering. Single Peakedness is a special case however, alternatives are required to be unidimensional and this is not always a reasonable assumption. Clearly, the biggest problem, that of intensity of preferences being ignored by voting systems, has not been properly addressed here nor indeed in the theory and practice of decision making. I feel it is lateral thinking in this and other directions in Social Choice Theory which should be highlighted, especially in view of Arrow's nihilistic findings.

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